# Efficient quantification of non-Gaussian spin distributions

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We study theoretically and experimentally the quantification of non-Gaussian distributions via non-destructive measurements. Using the theory of cumulants, their unbiased estimators, and the uncertainties of these estimators, we describe a quantification which is simultaneously efficient, unbiased by measurement noise, and suitable for hypothesis tests, e.g., to detect non-classical states. The theory is applied to cold $^{87}$Rb spin ensembles prepared in non-gaussian states by optical pumping and measured by non-destructive Faraday rotation probing. We find an optimal use of measurement resources under realistic conditions, e.g., in atomic ensemble quantum memories.

*Introduction* - Non-Gaussian states are an essential requirement for universal quantum computation [1, 2] and several quantum communication tasks with continuous variables, including improving the fidelity of quantum teleportation [3] and entanglement distillation [4, 5]. Optical non-Gaussian states have been demonstrated [6–10] and proposals in atomic systems [11–14] are being actively pursued. In photonic systems, histograms [15] and state tomography [6, 7, 9, 10] have been used to show non-Gaussianity, but require a large number of measurements. For material systems with longer time-scales these approaches may be prohibitively expensive. Here we demonstrate the use of cumulants, global measures of distribution shape, to show non-Gaussianity in an atomic spin ensemble. Cumulants can be used to show non-classicality [16–18], can be estimated with few measurements and have known uncertainties, a critical requirement for proofs of non-classicality.

*Approach* - Quantification or testing of distributions has features not encountered in quantification of observables. For example, experimental measurement noise appears as a distortion of the distribution that cannot be "averaged away" by additional measurements. As will be discussed later, the theory of cumulants naturally handles this situation. We focus on the fourth-order cumulant $\kappa_4$, the lowest-order indicator of non-Gaussianity in symmetric distributions such as Fock [19] and "Schrödinger kitten" states [7, 11]. We study theoretically and experimentally the noise properties of Fisher's unbiased estimator of $\kappa_4$, i.e., the fourth "k-statistic" $k_4$, and find optimal measurement conditions. Because $\kappa_4$ is related to the negativity of the Wigner function [16], this estimation is of direct relevance to detection of non-classical states. We employ quantum non-demolition measurement, a key technique for generation and measurement of non-classical states in atomic spin ensembles [20, 21] and nano-mechanical oscillators [22].

*Moments, cumulants and estimators* - A continuous random variable $X$ with probability distribution function $P(X)$ is completely characterized by its moments $\mu_k \equiv \int X^k P(X) dX$ or cumulants $\kappa_n = \mu_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \mu_{n-k} \kappa_k$, where $\binom{n}{k}$ is the binomial coefficient.

Since Gaussian distributions have $\kappa_{n>2} = 0$, estimation of $\kappa_4$, (or $\kappa_3$ for non-symmetric distributions), is a natural test for non-Gaussianity. In an experiment, a finite sample $\{X_1 \ldots X_N\}$ from $P$ is used to estimate the $\kappa$'s. Fisher's unbiased estimators, known as "k-statistics" $k_n$, give the correct expectation values $\langle k_n \rangle = \kappa_n$ for finite $N$ [23]. Defining $S_n = \sum_i X_i^n$ we have:

$$k_3 = (2S_1^3 - 3NS_1S_2 + N^2S_3)/N_{(2)} \tag{1}$$
$$k_4 = \left(-6S_1^4 + 12NS_1^2S_2 - 3N(N-1)S_2^2 \right.$$
$$\left. -4N(N-1)S_1S_3 + N^2(N+1)S_4\right)/N_{(3)} \tag{2}$$

where $N_{(m)} \equiv N(N-1)\ldots(N-m)$ .

We need the uncertainty in the cumulant estimation to test for non-Gaussianity, or to compare non-Gaussianity between distributions. For hypothesis testing and maximum-likelihood approaches, we need the variances of $k_3, k_4$ for a given $P$. These are found by combinatorial methods and given in reference [23]:

$$\mathrm{var}(k_3) = \kappa_6/N + 9N(\kappa_2\kappa_4 + \kappa_3^2)/N_{(1)} + 6N^2\kappa_2^3/N_{(2)} \tag{3}$$
$$\mathrm{var}(k_4) = \kappa_8/N + 2N(8\kappa_6\kappa_2 + 24\kappa_5\kappa_3 + 17\kappa_4^2)/N_{(1)}$$
$$+72N^2(\kappa_4\kappa_2^2 + 2\kappa_3^2\kappa_2)/N_{(2)}$$
$$+24N^2(N+1)\kappa_2^4/N_{(3)}. \tag{4}$$

It is also possible to estimate the uncertainty in $k_4$ from data $\{X\}$ using estimators of higher order cumulants [23]. The efficiency of cumulant estimation is illustrated in Fig. 1.

*Measurement noise* - When the measured signal is $Z = X + Y$, where $X$ is the true value and $Y$ is uncorrelated noise, the measured distribution is the convolution $P(Z) = P(X) \otimes P(Y)$. The effect of this distortion on cumulants is the following: for independent variables, cumulants accumulate (i.e., add) [23], so that
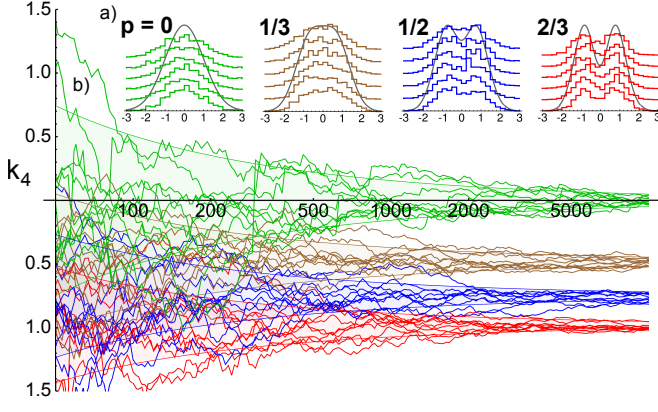
FIG. 1. (color online) Simulated estimator $k_4$ as a function of sample size $N$. a) (insets) black curves show quadrature distributions of states $\rho = (1-p)\,|0\rangle\,\langle 0| + p\,|1\rangle\,\langle 1|$, scaled to unit variance, and six $N = 1000$ histograms (offset for clarity) for $p = 0$ (green), $1/3$ (brown), $1/2$ (blue) and $2/3$ (red). b) Ten realizations of $k_4$ versus $N$ drawn from each of the four distributions. Shaded regions show $\kappa_4 \pm \sqrt{\mathrm{var}(k_4)}$, from Eqs (2), (4). With $N = 1000$, $k_4$ distinguishes $p = 1/2$ (blue) from $p = 0$ (green, Gaussian) with $> 7\sigma$ significance, even though the histograms look similar "to the eye."

$\kappa_n^{(Z)} = \kappa_n^{(X)} + \kappa_n^{(Y)}$, where $\kappa_n^{(Q)}, k_n^{(Q)}$ indicate $\kappa_n, k_n$ for distribution $P(Q)$. The extremely important case of uncorrelated, zero-mean Gaussian noise, $\kappa_2^{(Y)} = \sigma_Y^2$ and other cumulants zero, is thus very simple: $\kappa_n^{(Z)} = \kappa_n^{(X)}$ except for $\kappa_2^{(Z)} = \kappa_2^{(X)} + \sigma_Y^2$. Critically, added Gaussian noise does not alter the observed $\kappa_3, \kappa_4$.

*Experimental system and state preparation* - We test this approach by estimating non-Gaussian spin distributions in an atomic ensemble, similar to ensemble systems being developed for quantum networking with non-Gaussian states [24]. The collective spin component $F_z$ is measured by Faraday rotation using optical pulses. The detected Stokes operator is $S_y^{(\mathrm{out})} = S_y^{(\mathrm{in})} + GN_{\mathrm{L}}F_z/2$, where $G$ is a coupling constant, $N_{\mathrm{L}}$ is the number of photons, and $S_y^{(\mathrm{in})}$ is the input Stokes operator, which contributes quantum noise. In the above formulation $X = F_z$, $Y = 2S_y^{(\mathrm{in})}/(GN_{\mathrm{L}})$ and $Z = 2S_y^{(\mathrm{out})}/(GN_{\mathrm{L}})$.

The experimental system is described in detail in references [21, 25, 26]. An ensemble of $\sim 10^6$ $^{87}$Rb atoms is trapped in an elongated dipole trap made from a weakly focused 1030 nm beam and cooled to 25 µK. A non-destructive measurement of the atomic state is made using pulses of linearly polarized light detuned 800 MHz to the red of the $F = 1 \rightarrow F' = 0$ transition of the D$_2$ line and sent through the atoms in a beam matched to the transverse cloud size. The pulses are of 1 µs duration, contain $3.7 \times 10^6$ photons on average, and are spaced by 10 µs to allow individual detection. The 240:1 aspect ratio of the atomic cloud creates a strong paramagnetic Faraday interaction

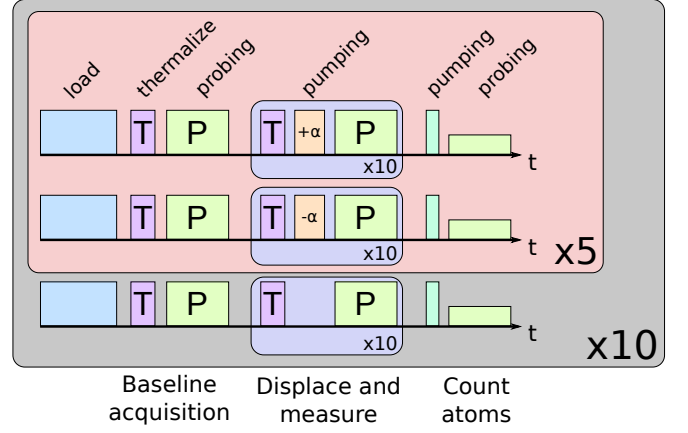$G \approx 6 \times 10^{-8}$ rad/spin. After interaction with the



FIG. 2. Experimental sequence: The experimental sequence divides into distinct tasks. Baseline acquisition: prepare the thermal state and probe to measure the residual rotation. Displace and measure (DM[$\alpha$]): prepare the thermal state, displace by $\alpha$ and probe. Thanks to atom loss at each thermalization, the atom number is varied by repeating DM several times. Measure number of atoms $N_A$: by pumping the atoms into F=1, $m_F$=1 and probing we measure the number of atoms in the trap. To correct for drifts, a sequence without displacement (DM[0]) is performed every 11 runs. We perform the sequence varying the displacement to acquire a dataset of quantum-noise-limited measurements of $P_\alpha^{(\mathrm{NG})}(S_y^{(\mathrm{out})})$ for different $\alpha$.

atoms, $S_y^{(\mathrm{out})}$ is detected with a shot noise limited (SNL) balanced polarimeter in the $\pm 45°$ basis. $N_{\mathrm{L}}$ is measured with a beam-splitter and reference detector before the atoms. The probing-plus-detection system is shot-noise-limited above $3 \times 10^5$ photons/pulse. Previous work with this system has demonstrated QND measurement of the collective spin $F_z$ with an uncertainty of $\sim 500$ spins [21, 26].

We generate Gaussian and non-Gaussian distributions with the following strategy: we prepare a "thermal state" (TS), an equal mixture of the $F = 1, m_F = -1, 0, 1$ ground states, by repeated unpolarized optical pumping between the $F = 1$ and $F = 2$ hyperfine levels, finishing in $F = 1$ [26]. By the central limit theorem, the TS of $10^6$ atoms is nearly Gaussian with $\langle F_z \rangle = 0$ and $\mathrm{var}(F_z) = \sigma^2 = 2N_A/3$. By optical pumping with pulses of circularly-polarized light we displace this to $\langle F_z \rangle = \alpha$, with negligible change in $\mathrm{var}(F_z)$ [27], to produce $P_\alpha(F_z) = (\sigma\sqrt{2\pi})^{-1} \exp[-(F_z - \alpha)^2/(2\sigma^2)]$. By displacing different TS alternately to $\alpha_+$ and $\alpha_-$, we produce an equal statistical mixture of the two displaced states, $P_\alpha^{(\mathrm{NG})}(F_z) = [P_{\alpha_+}(F_z) + P_{\alpha_-}(F_z)]/2$. With properly-chosen $\alpha_\pm$, $P_\alpha^{(\mathrm{NG})}(F_z)$ closely approximates marginal distributions of mixtures of $n = 0, 1$ Fock states and $m = N, N-1$ symmetric Dicke states. The experimental sequence is shown in Fig. 2.

*Detection, Analysis and Results* - For each preparation, 100 measurements of $F_z$ are made, with readings (i.e.,
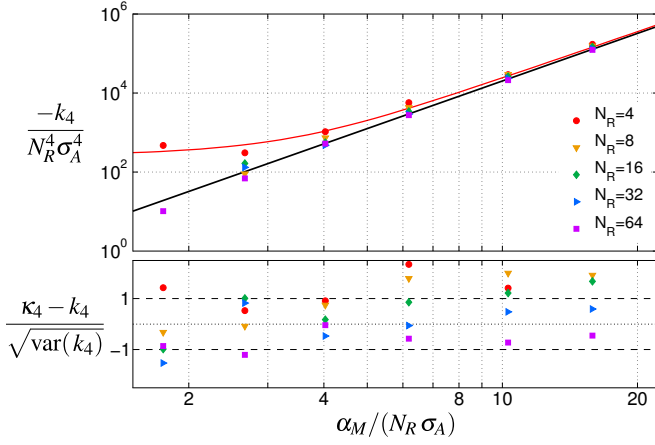
FIG. 3. (color online) Measured and predicted $k_4$ with residuals for non-Gaussian distributions of different $\alpha$. Readout noise is varied by the choice of $N_R$. Data is normalized to $N_R$ and $\sigma_A$. Top: Points show normalized $-k_4$ calculated from $N = 100$ preparations of the ensemble with different $\alpha$ (horizontal axis), and $N_R$ (colors). Black line indicates expected $-\kappa_4$, red line (top) shows $-\kappa_4 + \sqrt{\mathrm{var}(k_4)}$ calculated from the distribution parameters for the largest readout noise. Bottom: normalized residuals $(-k4 + \kappa_4)/\sqrt{\mathrm{var}(k_4)}$. The normalization is done with the expected $\mathrm{var}(k_4)$ for each $N_R$. Measured $k_4$ agrees well with theory, in particular, measurement noise increases the observed variance, but not the expectation.

estimated $F_z$ values by numerical integration of the measured signal) $m_i = 2S_y^{(\mathrm{out},i)}/N_L^{(i)}$. Because the measurement is non-destructive and shot noise limited, we can combine $N_R$ readings in a higher-sensitivity meta-pulse with reading $M \equiv \sum m_i$ [26]. This has the distribution $P_{\alpha_\pm}(M) = \exp[-(M-\alpha_\pm)^2/(2\sigma_M^2)]/(\sigma_M\sqrt{2\pi})$ where the variance $\sigma_M^2 = \sigma_A^2 N_A'^2 N_R^2 + \sigma_R^2$ includes atomic noise $\sigma_A^2 N_A'^2$ and readout noise, $\sigma_R^2 = N_R/N_L$ with $N_A' = N_A/N_A^{MAX}$. The variance $\sigma_A^2$ is determined from the scaling of $\mathrm{var}(M)$ with $N_A$ and $N_R$, as in [26]. The readout noise can be varied over two orders of magnitude by appropriate choice of $N_R$. For one probe pulse and the maximum number of atoms we have $\sigma_R^2/\sigma_A^2 = 84.7$.

To produce a non-Gaussian distribution, we compose metapulses from $N_R$ samples drawn from displaced thermal state (DM$[\alpha_+]$ or DM$[\alpha_-]$) preparations with equal probability, giving distribution $P_\alpha^{(\mathrm{NG})}(M) = [P_{\alpha_+}(M) + P_{\alpha_-}(M)]/2$. With $\alpha_M \equiv (\alpha_+ - \alpha_-)/2$, the distribution has $\kappa_{2n+1} = 0$, $\kappa_2 = \alpha_M^2 + \sigma_M^2$, $\kappa_4 = -2\alpha_M^4$, $\kappa_6 = 16\alpha_M^6$, $\kappa_8 = -272\alpha_M^8$. Our ability to measure the non-Gaussianity is determined by $\langle k_4 \rangle = \kappa_4$ and from Eq (4)

$$\mathrm{var}(k_4) = 136N\alpha_M^8/N_{(1)} - 144N^2\alpha_M^4(\alpha_M^2 + \sigma_M^2)^2/N_{(2)}$$
$$+24N^2(N+1)(\alpha_M^2 + \sigma_M^2)^4/N_{(3)}. \quad (5)$$

As shown in Fig. 3, the experimentally obtained values agree well with theory, and confirm the independence from measurement noise.
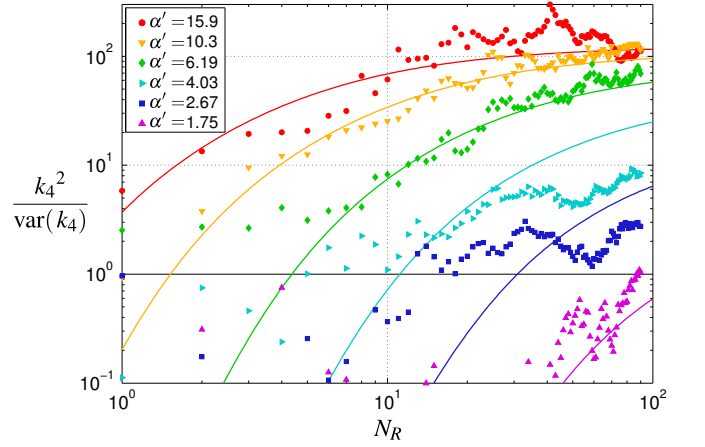


FIG. 4. (color online) Signal-to-noise in estimation of $\kappa_4$ versus readout noise for different $\alpha' = \alpha_M/(N_R \sigma_A)$. Points show measurement results, lines show theory. (details in the text)

The "signal-to-noise ratio" for $\kappa_4$, $S = \kappa_4^2/\mathrm{var}(k_4)$, is computed using Eq. (5), $\kappa_4 = -2\alpha_M^4$, and experimental $\alpha_M, N_R, \sigma_R$, is shown as curves in Fig 4. We can confirm this $S$ experimentally by computing $S_N \equiv \langle k_4 \rangle^2/\mathrm{var}(k_4)$ using $k_4$ values derived from several realizations of the experiment, each sampling $P_\alpha^{NG}$ $N$ times. In the limit of many realizations $S_N \to S$. We employ a bootstrapping technique: From 100 samples of $P_\alpha^{(\mathrm{NG})}(M)$ for given parameters $\alpha_M, N_R$ and $N_A$, we derive thirty-three $N = 20$ realizations by random sampling without replacement, and compute $\langle k_4 \rangle$ and $\mathrm{var}(k_4)$ on the realizations. As shown in Fig. 4, good agreement with theory is observed.

*Optimum estimation of non-Gaussian distributions* - Finally, we note that in scenarios where measurements are expensive relative to state preparation (as might be the case for QND measurements of optical fields or for testing the successful storage of a single photon in a quantum memory), optimal use of measurement resources (e.g. measurement time) avoids both too few preparations and too few probings.

We consider a scenario of practical interest for quantum networking: a heralded single-photon state is produced and stored in an atomic ensemble quantum memory. Assuming the ensemble is initially polarized in the $\hat{X}$ direction, the storage process maps the quadrature components $X, P$ onto the corresponding atomic spin operators $X_A, P_A \propto F_z, -F_y$, respectively. QND measurements of $F_z$ are used to estimate $X_A$, and thus the non-Gaussianity of the stored single photon. Due to imperfect storage, this will have the distribution of a mixture of $n = 0$ and $n = 1$ Fock states: $\rho = (1 - p)|0\rangle\langle 0| + p|1\rangle\langle 1|$. For a quadrature $X$, we have the following probability distribution $P_p(X) = \exp[-x^2/(2\sigma_0^2)]\left(px^2/\sigma_0^2 + 1 - p\right)/\left(\sqrt{2\pi}\sigma_0\right)$, where $\sigma_0$ is the width of the $n = 0$ state.

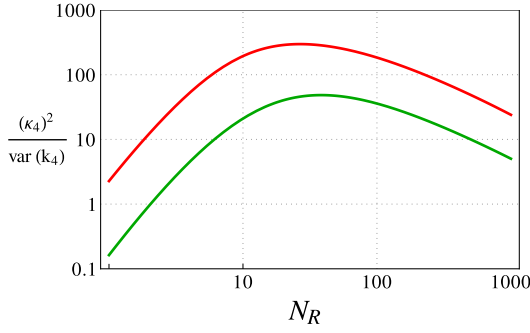Taking in account the readout noise $\sigma_R^2$, the cumulants

FIG. 5. (color online) Signal-to noise-ratio $S$ versus $N_R$ for a fixed probe number $N_M N_R = 1 \times 10^5$ for the probability distribution associated with Fock state mixture described in the text with a normalized n=0 width $\sigma_0 = 1$. Red curve (top): p=1. Green curve (bottom): p= 0.5 with SNL measurement: $\sigma_R = \sqrt{20/N_R}$.

are $\kappa_{\text{odd}} = 0$, $\kappa_2 = (2p+1)\sigma_H^2 + \sigma_R^2$, $\kappa_4 = -12p^2\sigma_H^4$, $\kappa_6 = 240p^3\sigma_H^6$, $\kappa_8 = -10080p^4\sigma_H^8$, where the readout noise $\sigma_R^2$ is included as above. Here $\kappa_4$ is directly related to the classicality of the state, since $p > 0.5$ implies a negative Wigner distribution [19].

For a fixed total number of measurement resources $N_M N_R$, an optimal distribution of resources per measurement $N_R$ exists as shown in Fig. 5. With increasing $N_R$, the signal-to-noise first increases due to the improvement of the measurement precision. Then, once the increased measurement precision no longer gives extra information about $k_4$, the precision decreases due to reduced statistics because of the limited total number of probes. For a large total number of measurements, we can derive a simplified expression of this optimum. We derive asymptotic expressions of $S$: $S_L$ ($S_H$) for $\sigma_R \ll \sigma_0$ ($\sigma_R \gg \sigma_0$). The optimal $N_R$ is found by solving $S_L = S_R$ giving $\sigma_R^8 \approx \sigma_0^8(1 + 8p - 12p^2 + 48p^3 - 24p^4)$. For this optimal $\sigma_R$, the measurement noise is in the same order of magnitude as the characteristic width of the non-Gaussian distribution.

*Conclusion* - The cumulant-based methods described here should be very attractive for experiments with non-Gaussian states of material systems such as atomic ensembles and nano-resonators, for which the state preparation time is intrinsically longer, and for which measurement noise is a greater challenge than in optical systems. Cumulant-based estimation is simultaneously efficient, requiring few preparations and measurements, accommodates measurement noise in a natural way, and facilitates statistically-meaningful tests, e.g., of non-classicality. Experimental tests with a cold atomic ensemble demonstrate the method in a system highly suitable for quantum networking, while the theory applies equally to other quantum systems of current interest.

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